

INVARIANT DIFFERENTIAL OPERATORS IN POSITIVE CHARACTERISTIC

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ABSTRACT. We consider an analog of the problem Veblen formulated in 1928 at the IMC: classify invariant differential operators between “natural objects” (spaces of either tensor fields, or jets, in modern terms) over a real manifold of any dimension. For unary operators, the problem was solved by Rudakov (no nonscalar operators except the exterior differential); for binary ones, by Grozman (there are no operators of orders higher than 3, operators of order 2 and 3 are, bar an exception in dimension 1, compositions of order 1 operators which, up to dualization and permutation of arguments, form 8 families). In dimension one, Grozman discovered an indecomposable selfdual operator of order 3 that does not exist in higher dimensions. We solve Veblen’s problem in the 1-dimensional case over the ground field of positive characteristic. In addition to analogs of the Berezin integral (strangely overlooked so far) and binary operators constructed from them, we discovered two more (up to dualization) types of indecomposable operators of however high order: analogs of the Grozman operator and a completely new type of operators.

1. INTRODUCTION

In the representation theory of a given algebra or group, the usual ultimate goal is a clear description of its irreducible modules to begin with, indecomposable ones at the next step. Other goals are sometimes also natural and reachable, cf. those we discuss here and in review [BF].

1.1. Discussion of open problems. The history of mathematics shows that problems with a nice (at least in some sense, for example, short) answer often turn out to be “reasonable”. I. Gelfand used to say that, the other way round, easy-to-formulate lists are answers to “reasonable” problems and advised to try to formulate such problems once we got a short answer.

Hereafter, \mathbb{K} is any field of characteristic $p > 0$, unless otherwise specified.

Speaking of Lie algebras or superalgebras, and their representations over \mathbb{K} , which of them is it natural to consider? In [LL], Deligne suggests that one should begin with restricted ones: unlike nonrestricted ones, the restricted Lie (super)algebras correspond to groups, in other words: to geometry. Certain problems concerning nonrestricted algebras are also natural (although tough), and have a short answer, e.g., classification of simple Lie algebras for $p > 3$ (for its long proof, see [S, BGP]).

Rudakov and Shafarevich [RSh] were the first to describe ALL, not only restricted, irreducible representations of $\mathfrak{sl}(2)$ for $p > 2$. Dolotkazin solved the same problem for $p = 2$, see [Do]; more precisely, he described the irreducible representations of the simple 3-dimensional Lie algebra $\mathfrak{o}^{(1)}(3)$. The difference of Dolotkazin’s problem from that considered in [RSh] is that, unlike $\mathfrak{sl}(2)$ for $p > 2$, the Lie algebra $\mathfrak{o}^{(1)}(3)$ is not restricted. These two results show that the description of all irreducible representations looks feasible, to an extent, at least for Lie (super)algebras with indecomposable Cartan matrix or their simple “relatives” (for the classification of both types of (super)algebras over \mathbb{K} , see [BGL]).

1991 *Mathematics Subject Classification.* Primary 17B50, 17B66, 17B56.

Key words and phrases. Veblen’s problem, invariant differential operator, positive characteristic.

The problem we are solving here for $p > 0$ resembles its super version **over** \mathbb{C} ; we recall the known results. Simple Lie (super)algebras of vector fields with polynomial or formal coefficients constitute another class of interesting Lie (super)algebras. **Over** \mathbb{C} , they are classified and their continuous (in the x -adic topology) irreducible representations with a vacuum (highest or lowest) weight vector are classified as well in the following cases:

- For the four series of simple Lie algebras, and their direct super versions, the answer is as follows, see the review [GLS]: the module $T(V)$ of tensor fields coinduced from any irreducible module V with lowest weight vector over the subalgebra of linear vector fields is irreducible, unless $T(V) \simeq \Omega^k$, the module of exterior k -forms.

Rudakov proved that the exterior differential is the only (nonscalar) operator between the modules of exterior differential forms invariant with respect to all changes of indeterminates, or, equivalently, for the general vectorial Lie algebra. Together with the Poincaré lemma (stating that the sequence (1) is exact over any star-shaped domain) this yields a complete description of continuous irreducible modules with lowest weight.

- For the remaining 3 series of simple vectorial Lie algebras (and analogous simple Lie superalgebras **in their standard grading**) the answer is essentially the same.

- For simple vectorial Lie superalgebras in their **nonstandard** grading (listed in [LS]), and for the modules of tensor fields with **infinite-dimensional fibers** (i.e., $\dim V = \infty$) even over finite-dimensional simple vectorial Lie superalgebras, many never previously known differential operators appear, see [GLS, Lint].

- In the purely odd case the Berezin integral has to be added to the list of invariant differential operators, see eq. (3).

- **Over** \mathbb{K} , it is tempting to conjecture that, at least for p “sufficiently large”, the description of irreducible modules over vectorial Lie algebras (at least \mathbb{Z} -graded, and “close to simple” ones) the description of irreducible modules would resemble that given by Rudakov over \mathbb{C} . For irreducible restricted modules over restricted simple vectorial Lie algebras of the 4 classical series and $p \geq 3$, this is almost so, i.e., the spaces of differential k -forms are the only reducible spaces of tensor fields, as proven by Krylyuk, see [Kry], although the complex of exterior differential forms is not exact, see Theorem 2.2.1.

There are, however, natural filtered deformations of simple nonrestricted vectorial Lie algebras; for $p \leq 5$, there are new types of simple vectorial Lie algebras and **no description of their irreducible modules is known; in particular, no description of intertwining operators between modules of tensor fields — the most natural ones over these Lie algebras.** Description of this type has a history.

1.1.1. History: Veblen’s problem. In 1928, at the IMC, O. Veblen formulated a problem which was later reformulated in more comprehensible terms by A. Kirillov and further reformulated as a purely algebraic problem by J. Bernstein who interpreted Rudakov’s solution of Veblen’s problem for unary operators; for setting in modern words and review, in particular, for a superization of Veblen’s problem, see [GLS].

Let $\mathbf{vect}(m)$ be the Lie algebra of polynomial vector fields (over a ground field of characteristic 0, say \mathbb{C}). Assuming the notion of the $\mathbf{vect}(m)$ -module $T(V)$ of (formal) tensor fields of type V , where V is a $\mathfrak{gl}(m)$ -module with lowest weight vector, is known, see [GLS]. Let us briefly recall the results concerning the **nonscalar** unary and binary invariant differential operators between spaces of tensor fields, although we only need the simplest version of spaces $T(V)$, namely, the spaces of weighted densities $\mathcal{F}_a := T(a \operatorname{tr})$, where tr is the 1-dimensional $\mathfrak{gl}(m)$ -module given by the trace (supertrace for $\mathfrak{gl}(m|k)$) and a is in the ground field. We assume that modules $T(V)$ in what follows are coinduced from **irreducible** modules V ; this is certainly so for modules \mathcal{F}_a . (In Veblen’s problem for simple vectorial Lie algebras over \mathbb{C} , it suffices to consider only

irreducible modules V ; for its superization and over \mathbb{K} , one has to consider indecomposable modules V , but one has to begin with irreducible ones, anyway.)

1.1.1a. Unary operators. The only unary $\mathbf{vect}(m)$ -invariant differential operators $T(V) \rightarrow T(W)$ are the exterior differential operators (so, practically, there is just one such operator) in the de Rham complex, where $\Omega^i = T(E^i(\text{id}))$, E^i is the functor of raising to the i th exterior power and id is the tautological m -dimensional $\mathbf{gl}(m)$ -module:

$$(1) \quad 0 \rightarrow \mathbb{C} \rightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega^{m-1} \xrightarrow{d} \Omega^m \rightarrow 0.$$

In super setting, where $\text{sdim}(\text{id}) = m|k$, the analog of the complex (1) is infinite without the “top term”

$$(2) \quad 0 \rightarrow \mathbb{C} \rightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots$$

its dualization brings the complex of *integrable* forms which **in the purely odd case of superdimension** $0|k$ terminates with an order- k **differential** operator called the *Berezin integral*:

$$(3) \quad \begin{aligned} \dots \xrightarrow{d} \Sigma_{-1}(m|k) &\xrightarrow{d} \Sigma_0(m|k) \xrightarrow{d} 0 && \text{for } m \neq 0, \\ \dots \xrightarrow{d} \Sigma_{-1}(0|k) &\xrightarrow{d} \Sigma_0(0|k) \xrightarrow{f} \mathbb{K} \rightarrow 0 && \text{for } m = 0. \end{aligned}$$

Over \mathbb{C} , the de Rham complex (2) is exact, while the complex of integrable forms (3) has 1-dimensional cohomology in the $(-m)$ th term; the cohomology is spanned, see [BL, MaGF, Del], by (here the u_i are the even indeterminates, the θ_j are the odd ones for clarity; in Notation 2.1 all indeterminates are called u)

$$(4) \quad \theta_1 \dots \theta_k \partial_{u_1} \dots \partial_{u_m} \text{vol}(u|\theta)$$

and the super version of the volume element $\text{vol}(u) = du_1 \wedge \dots \wedge du_m$ is the class $\text{vol}(u|\theta)$ of the element

$$(5) \quad du_1 \wedge \dots \wedge du_m \otimes \partial_{\theta_1} \wedge \dots \wedge \partial_{\theta_k}$$

in the indecomposable $\mathbf{gl}(V) = \mathbf{gl}(m|k)$ -module generated by the element (5) modulo the codimension 1 submodule, see [Del, LSoS].

1.1.1b. Binary operators. These are classified by P. Grozman, see [GInv]. Recall that the formal dual of $T(V)$ on the m -dimensional space is not $T(V^*)$ but $T(V^* \otimes \text{tr})$. Hence, with every unary operator $D: T(V) \rightarrow T(W)$ there exists its dual $D^*: T(W^* \otimes \text{tr}) \rightarrow T(V^* \otimes \text{tr})$.

In particular, on the line, with every operator $D: \mathcal{F}_a \rightarrow \mathcal{F}_{a+\deg D}$ of order $\deg D$, there exists its dual $D^*: \mathcal{F}_{1-a-\deg D} \rightarrow \mathcal{F}_{1-a}$ of the same order.

Similarly, with every binary operator $D: \mathcal{F}_a \otimes \mathcal{F}_b \rightarrow \mathcal{F}_{a+b+\deg D}$, there exist the two its duals $D^{*1}: \mathcal{F}_{1-a-b-\deg D} \otimes \mathcal{F}_b \rightarrow \mathcal{F}_{1-a}$ and $D^{*2}: \mathcal{F}_a \otimes \mathcal{F}_{1-a-b-\deg D} \rightarrow \mathcal{F}_{1-b}$, as well as the operator $r(D): \mathcal{F}_b \otimes \mathcal{F}_a \rightarrow \mathcal{F}_{a+b+\deg D}$, given by the interchange of the arguments

$$r(D)(X, Y) := D(Y, X) \text{ for any } X \in \mathcal{F}_b \text{ and } Y \in \mathcal{F}_a.$$

Summary: Up to dualizations and interchange of the arguments, there are 8 series of invariant operators of order 1, all invariant differential operators of order 2 and 3 are compositions of order-1 operators, except for the case $m = 1$, where there is an indecomposable operator Gz of order 3 discovered by Grozman; for explicit expressions, see [GInv] and Subsec. 2.4. There are no operators of order > 3 .

Off-topic: 1) Among all 8 series of invariant operators of order 1, there are 4 series that determine an associative algebra and 3 series determine a Lie (super)algebra structure on their domain of definition. These Lie (super)algebras are either simple or simple modulo center or

contain a simple algebra of codimension 1: the corresponding products are the Poisson bracket, the Schouten bracket (more popular now under the name anti-bracket) and a deformed Schouten bracket, see [GInv, LSh].

2) Kirillov gave an example of the symbol of an invariant with respect to all coordinate changes *nonlocal* bilinear operator on the circle but did not identify the operator itself; in [IoMo] this operator is described in terms of the integral=residue; it is the only nonlocal invariant bilinear operator on the circle. In characteristic $p > 0$, same as on superpoints, analogs of various operators nonlocal over \mathbb{R} become local because if the space of functions is finite-dimensional, then any operator in this space can be viewed as differential, i.e., composition of derivations and multiplications by some functions.

1.1.1c. Invariant differential operators of higher arity. The only k -ary invariant operators classified so far are antisymmetric ones on \mathbb{C} , see [FeFu], and ternary operators between the spaces of weighted densities on \mathbb{C}^n , see [Bj].

1.2. Main result. The classification (Theorem 2.6) of binary $\mathbf{vect}(1; \underline{N})$ -invariant operators between modules of weighted densities; in particular, we discovered three series of indecomposable operators of however high order: 1) related to the analog of Berezin integral, 2) an analog of the Grozman operator, and 3) a completely new series of operators, Bj .

1.3. Open problems. 1) We conjecture that the Grozman operator has something to do with the contact structure since $\mathbf{vect}(1) = \mathfrak{k}(1)$. Over \mathbb{C} , it seems feasible to classify binary $\mathfrak{k}(2n+1)$ -invariant differential operators for $n > 0$ and complete the partial result for binary $\mathfrak{h}(2n)$ -invariant differential operators, solved only for $n = 1$, see [Ghm].

2) The only feasible super version of the problem we consider here is, it seems, classification of the $\mathfrak{k}(1; \underline{N}|k)$ -invariant operators for $k = 1$ or 2 for which the coinduced modules with lowest weight vectors are of the form \mathcal{F}_a . For the other pairs m, k the analog of Veblen's problem does not seem to be feasible.

3) Prove Conjecture 2.3.1.

2. VEBLEN'S PROBLEM OVER FIELDS OF CHARACTERISTIC $p > 0$

2.1. Notation. For an $m + k$ -tuple of nonnegative integers $\underline{r} = (r_1, \dots, r_a)$, where $r_i = 0$ or 1 for $i > m$, we introduce parity by setting $p(u_i) = \bar{0}$ for $i \leq m$ and $p(u_i) = \bar{1}$ for $i > m$. Set further

$$u_i^{(r_i)} := \frac{x_i^{r_i}}{r_i!} \quad \text{and} \quad u^{(\underline{r})} := \prod_{1 \leq i \leq a} u_i^{(r_i)}.$$

The idea is to formally replace fractions with $r_i!$ in denominators by inseparable symbols $u_i^{(r_i)}$ which are well-defined over fields of characteristic p because the structure constants are integers one can consider modulo p , in particular, if $k = 0$, the factor in the first parentheses below is equal to 1:

$$u^{(\underline{r})} \cdot u^{(\underline{s})} = \left(\prod_{m+1 \leq i \leq n} \min(1, 2 - r_i - s_i) \cdot (-1)^{\sum_{m < i < j \leq a} r_j s_i} \right) \cdot \binom{\underline{r} + \underline{s}}{\underline{r}} u^{(\underline{r} + \underline{s})},$$

where $\binom{\underline{r} + \underline{s}}{\underline{r}} := \prod_{1 \leq i \leq m} \binom{r_i + s_i}{r_i}.$

For an $m + k$ -tuple of positive integers $\underline{N} = (N_1, \dots, N_m, 1, \dots, 1)$, set $\underline{N}_{ev} = (N_1, \dots, N_m)$. The following supercommutative superalgebra is the analog of the algebra of functions when $p > 0$:

$$\mathcal{O}(m; \underline{N}_{ev}|k) := \mathbb{K}[u; \underline{N}] := \text{Span}_{\mathbb{K}}(u^{(\underline{r})} \mid r_i < p^{N_i} \text{ for } i \leq m, 0 \leq r_i \leq 1 \text{ for } i > m).$$

The Lie superalgebra $\mathbf{vect}(m; \underline{N}|k)$ of its *distinguished* derivations consists of the vector fields of the form $\sum f_i(u)\partial_i$, where $f_i \in \mathcal{O}(m; \underline{N}_{ev}|k)$ and ∂_i are *distinguished* partial derivatives defined by the condition $\partial_j(u_i^{(r_i)}) = \delta_{ji}u_i^{(r_i-1)}$.

2.2. Unary operators: De Rham complex for $k = 0$ (no super).

2.2.1. Theorem ([Kry]). Set

$$\tau(\underline{N}) = (p^{N_1} - 1, \dots, p^{N_m} - 1).$$

Denote: $Z^i(m; \underline{N}) := \{\omega \in \Omega^i(m; \underline{N}) \mid d\omega = 0\}$ and $B^i(m; \underline{N}) := \{d\omega \mid \omega \in \Omega^{i-1}(m; \underline{N})\}$.

On the m -dimensional space over \mathbb{K} , the sequence

$$0 \longrightarrow \mathbb{K} \longrightarrow \Omega^0(m; \underline{N}) \xrightarrow{d} \Omega^1(m; \underline{N}) \xrightarrow{d} \Omega^2(m; \underline{N}) \xrightarrow{d} \dots \xrightarrow{d} \Omega^m(m; \underline{N}) \xrightarrow{d} 0$$

is not exact: the space $H^a(m; \underline{N}) := Z^a(m; \underline{N})/B^a(m; \underline{N})$ is spanned by the elements

$$u_{i_1}^{(\tau(\underline{N})_{i_1})} \dots u_{i_k}^{(\tau(\underline{N})_{i_k})} du_{i_1} \dots du_{i_k} \text{ where } a = i_1 + \dots + i_k.$$

In other words, the supercommutative superalgebra $H^\bullet(m; \underline{N})$ is generated by $H^1(m; \underline{N})$.

Proof. Induction on m . For $m = 1$, this is obvious. □

2.2.2. Corollary. There is an invariant differential operator of order $|\tau(\underline{N})| := (\sum p^{N_i}) - m$:

$$\Omega^m(m; \underline{N}) \xrightarrow{\int} \mathbb{K};$$

this \int is a, depending on \underline{N} , analog of the Berezin integral defined as follows:

$$\int f(u) \text{vol}(u) = \text{the coefficient of } u^{\tau(\underline{N})} = u_1^{(\tau(\underline{N})_1)} \dots u_m^{(\tau(\underline{N})_m)}, \text{ where } \text{vol}(u) = du_1 \wedge \dots \wedge du_m.$$

2.2.2a. Remark. We did not find the statement of the above Corollary in the literature; so formulation of the fact that there exists an integral depending on \underline{N} seems to be a new result.

2.3. Unary operators: De Rham complex for $k \neq 0$, i.e., in super setting. On the $m|k$ -dimensional space over \mathbb{K} , the de Rham complex goes ad infinitum

$$0 \longrightarrow \mathbb{K} \longrightarrow \Omega^0(m; \underline{N}|k) \xrightarrow{d} \Omega^1(m; \underline{N}|k) \xrightarrow{d} \Omega^2(m; \underline{N}|k) \xrightarrow{d} \dots$$

The dual to (2.3) complex of *integrable* (Deligne calls them *integral*) forms is as follows, where $\Sigma_0 := \Omega^0 \text{vol}(u|\theta)$:

$$(6) \quad \dots \xrightarrow{d} \Sigma_{-1}(m; \underline{N}|k) \xrightarrow{d} \Sigma_0(m; \underline{N}|k) \xrightarrow{\int} \mathbb{K} \longrightarrow 0.$$

Observe that, for $p > 0$, the integral is defined for $m \neq 0$ as well, whereas for $p = 0$, there is no such an invariant *differential* operator.

2.3.1. Conjecture. The only $\mathbf{vect}(m; \underline{N}|k)$ -invariant unary operators between modules of tensor fields coinduced from irreducible $\mathfrak{gl}(m|k)$ -modules are the exterior operator d and the Berezin integral \int .

2.4. The binary differential operators over \mathbb{C} for $m = 1$ (recapitulation, see [GInv]). For $m = 1$, we have $\mathcal{F}_a := \{f(u)(du)^a \mid f(u) \in \Omega^0 = \mathcal{F}_0\}$ on which the Lie derivative acts as follows:

$$(7) \quad L_D(f(u)(du)^a) = (D(f) + afg')(du)^a \text{ for any } D = g(u)\partial_u \in \mathbf{vect}(1).$$

For any bilinear differential operator

$$D: \mathcal{F}_a \otimes \mathcal{F}_b \longrightarrow \mathcal{F}_{a+b+\deg D}, \quad f(du)^a \otimes g(du)^b \longmapsto D(f, g)(du)^{a+b+\deg D},$$

it suffices to indicate (a, b) and $D(f, g)$. The operators are listed up to proportionality.

2.4.1. Order 1 operators. For generic values (a, b) , the invariant operators form a 1-dimensional space. For $(a, b) = (0, 0)$, and only in this case, we have a 2-dimensional space of operators.

(8)

(a, b)	$D(f, g)$	comment
$(0, 0)$	$P_{00}: \alpha f'g + \beta fg'$	for any $\alpha, \beta \in \mathbb{C}$
(a, b)	$\{\cdot, \cdot\}_{P.B.} := afg' - bf'g$	Poisson bracket in two even indeterminates u and du
$(-1, -1)$	$\{\cdot, \cdot\} := fg' - f'g$	contact bracket of generating functions $f \longmapsto K_f := f \frac{d}{du}$

2.4.2. Order 2 operators.

(a, b)	$D(f, g)$	comment
$(0, b)$	$f'g' - bf''g$	$\{df, g(du)^b\}_{P.B.}$
$(a, 0)$	$afg'' - f'g'$	$\{f(du)^a, dg\}_{P.B.}$
$(a, -1 - a)$	$afg'' + (2a + 1)f'g' + (a + 1)f''g$	$(\{f(du)^a, g(du)^{-1-a}\}_{P.B.})'$

2.4.3. Order 3 operators.

(a, b)	$D(f, g)$
$(0, 0)$	$T_1(f, g) = \{f', g'\}$
$(-2, 0)$	$T_1^{*1}(f, g) = f'g'' + 3f''g' + 2f'''g$
$(0, -2)$	$T_1^{*2}(f, g) = f''g' + 3f'g'' + 2fg'''$
$(-\frac{2}{3}, -\frac{2}{3})$	$Gz(f, g) = 2 \det \begin{pmatrix} f & g \\ f''' & g''' \end{pmatrix} + 3 \det \begin{pmatrix} f' & g' \\ f'' & g'' \end{pmatrix}$

2.5. New binary differential operators: $p > 0$ and $m = 1$. For a fixed $\underline{N} = N$, we list $\mathbf{vect}(1; \underline{N})$ -invariant operators up to interchange of arguments.

Recall that $\int: \Omega^1 = \mathcal{F}_1 \longrightarrow \mathbb{K} \subset \mathcal{F}_0$ is the order $p^N - 1$ operator, see Corollary 2.2.2 and $1 - p^N \equiv 1 \pmod{p}$. Therefore, for any $p > 0$ and \underline{N} , there are $\mathbf{vect}(1; \underline{N})$ -invariant bilinear operators acting in the spaces $\mathcal{F}_a \otimes \mathcal{F}_b$ in the following cases:

operator (a, b)	order	case
$\int \otimes \text{id}(1, a), (\int \otimes \text{id})^{*1}(1 - a, a), (\int \otimes \text{id})^{*2}(1, 1 - a)$	$k = p^N - 1$	1
$\int \otimes d = (\int \otimes d)^{*2}(1, 0), (\int \otimes d)^{*1}(0, 0), r(\int \otimes d)(0, 1)$	$k = p^N$	2
$\int \otimes \int = (\int \otimes \int)^{*1} = (\int \otimes \int)^{*2}(1, 1)$	$k = 2(p^N - 1)$	3

The operator $(\int \otimes \text{id})^{*1}(1 - a, a)$ is of the form (L for “long” expression)

$$(12) \quad L_{p,N}(f, g) = \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} (-1)^i (f^{(i)}g^{(k-i)} + f^{(k-i)}g^{(i)}) + \begin{cases} (-1)^{k/2} f^{(k/2)}g^{(k/2)} & \text{if } p > 2 \\ & \text{if } p = 2. \end{cases}$$

2.5.1. Lemma. For $(a, b) = (1, 1)$, there exists a $\mathbf{vect}(1; \underline{N})$ -invariant bilinear operator $Bj_{p,m}$ of order $k = p^m - 1$ for any $m \leq N$ given by the expression

$$(13) \quad Bj_{p,m}(f, g) = \det \begin{pmatrix} f & g \\ f^{(k)} & g^{(k)} \end{pmatrix}.$$

Its duals for $(a, b) = (1, 0)$ and $(0, 1)$, respectively, are as follows:

$$(14) \quad Bj_{p,m}^{*2}(f, g) = fg^{(k)} + \sum_{i=1}^{\lfloor (k-1)/2 \rfloor} (-1)^i (f^{(i)}g^{(k-i)} + f^{(k-i)}g^{(i)}) + \begin{cases} (-1)^{k/2} f^{(k/2)}g^{(k/2)} & \text{if } p > 2 \\ & \text{if } p = 2 \end{cases}$$

and

$$(15) \quad Bj_{p,m}^{*1}(f, g) = f^{(k)}g + \sum_{i=1}^{\lfloor (k-1)/2 \rfloor} (-1)^i (f^{(i)}g^{(k-i)} + f^{(k-i)}g^{(i)}) + \begin{cases} (-1)^{k/2} f^{(k/2)}g^{(k/2)} & \text{if } p > 2 \\ & \text{if } p = 2. \end{cases}$$

Proof. Every order k bilinear differential operator D is of the form

$$(f, g) \mapsto \sum_{i+j=k} \alpha_{i,j} f^{(i)}g^{(j)}, \text{ where } \alpha_{i,j} \in \mathbb{K}.$$

For N big enough (such that $p^N > \deg D + 1$) and a given pair (a, b) , the $\mathbf{vect}(1; \underline{N})$ -invariance of D implies the following system of equations

$$\left(a \binom{i}{r-1} + \binom{i}{r} \right) \alpha_{i,k-i} + \left(b \binom{k-i+r-1}{r-1} + \binom{k-i+r-1}{r} \right) \alpha_{i-r+1,k-i+r-1} = 0,$$

for $i = 1, \dots, k$ and $2 \leq r \leq k+1$. In the case of Lemma, $(a, b) = (1, 1)$; besides $\alpha_{i,j} = 0$ whenever $i + j \neq k$. The above system becomes

$$(16) \quad \binom{i+1}{r} \alpha_{i,k-i} + \binom{k-i+r}{r} \alpha_{i-r+1,k-i+r-1} = 0 \text{ for } i = 1, \dots, k \text{ and } 2 \leq r \leq k+1.$$

If $r = k+1$, then i must be equal to k , so we get the condition $\alpha_{0,k} + \alpha_{k,0} = 0$. If $r < k+1$, then the above equations are either identically zero (since $\alpha_{i,j} = 0$ whenever $i + j \neq k$) or of the form

$$\binom{k+1}{r} \alpha_{k,0} = 0 \text{ and } \binom{k+1}{r} \alpha_{0,k} = 0.$$

But $\binom{k+1}{r} = 0 \pmod{p}$, since $k+1 = p^N$.

The claim on dual operators is subject to a direct verification. \square

2.5.2. The transvectants (also known as Cohen-Rankin brackets). Recall that for any $p \neq 2$, there is an embedding $\mathfrak{sl}(2) \hookrightarrow \mathbf{vect}(1; \underline{N})$. (For $p = 2$, the analog of this embedding is the embedding $\mathfrak{o}^{(1)}(3) \hookrightarrow \mathbf{vect}(1; \underline{N})$ for $\underline{N} > 1$.) The *transvectants* are $\mathfrak{sl}(2)$ -invariant bilinear differential operators discovered by Gordan. As a matter of fact, there is a misprint in the explicit expressions of the transvectants $J_{a,b}^k$ in [BjOv]; the corrected expression is (up to a numerical factor)

$$(17) \quad J_{a,b}^k(f, g) = \sum_{0 \leq i \leq k} (-1)^i \binom{2a+k-1}{k-i} \binom{2b+k-1}{i} f^{(i)}g^{(k-i)}.$$

This expression is for generic a and b , and $p \neq 2$. For the exceptional values of a and b over \mathbb{C} , see [Bj2], Prop.2. In this paper, we are mainly interested in the explicit expressions of the transvectants for $p = 2$.

2.5.3. Lemma. For $(a, b) = (1, 1)$, there exists a $\mathbf{vect}(1; \underline{N})$ -invariant bilinear operator of order $k = p^m - 2$ for any $m \leq N$ if $p \neq 2$ whereas for $p = 2$ it should be any $m \leq N + 1$ (because of the term $f^{(k/2)}g^{(k/2)}$) given by the expression

$$(18) \quad G_{z_{p,m}}(f, g) = \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} (-1)^i \det \begin{pmatrix} f^{(i)} & g^{(i)} \\ f^{(k-i)} & g^{(k-i)} \end{pmatrix} + \begin{cases} f^{(k/2)}g^{(k/2)} & \text{if } p = 2 \\ \text{otherwise.} \end{cases}$$

This operator coincides with the Grozman operator if $p = 5$ and $m = 1$. Observe that Gz is selfdual over any field it is defined, but $Gz \circ (d \otimes d)$ is not selfdual.

Proof. Let, for simplicity, $p \neq 2$ (for $p = 2$ the proof is more or less the same). Since $(a, b) = (1, 1)$, the invariance with respect to $\mathbf{vect}(1; \underline{N})$ is equivalent to the system (16). For $i = k$ or $i = 0$, we have $\alpha_{k,0} + \alpha_{0,k} = 0$ which is certainly satisfied. Now, for $i \neq 0, k$, the system becomes

$$\binom{i+1}{r}(-1)^i + \binom{k-i+r}{r}(-1)^{i-r+1} = 0 \text{ for } i = 1, \dots, k \text{ and } 2 \leq r \leq k+1.$$

To show that the system is satisfied, we proceed by induction. If $i = 1$, then r must be equal to 2. The result follows since

$$\binom{k+1}{2} = \binom{p^m-1}{2} = 1 \pmod{p}.$$

Now suppose that the equality is true at i for every r such that $r \leq i+1$. It follows that

$$\begin{aligned} \binom{k-(i+1)+r}{r} &= \binom{k-i+r}{r} - \binom{k-i+r-1}{r-1} \text{ ‘Pascal’s triangle’} \\ &= (-1)^r \binom{i+1}{r} - (-1)^{r-1} \binom{i+1}{r-1} \pmod{p} \text{ induction hypothesis} \\ &= (-1)^r \left(\binom{i+1}{r} + \binom{i+1}{r-1} \right) \pmod{p} \\ &= (-1)^r \binom{i+2}{r} \pmod{p} \text{ ‘Pascal’s triangle’} \quad \square \end{aligned}$$

2.6. Theorem. Up to the interchange of arguments, the indecomposable $\mathbf{vect}(1; \underline{N})$ -invariant bilinear operators $D: \mathcal{F}_a \otimes \mathcal{F}_b \rightarrow \mathcal{F}_c$ are only those of the form (11)–(18). For the complete list of $\mathbf{vect}(1; \underline{N})$ -invariant bilinear operators of order ≤ 7 , see tables (19)–(26).

Although we have found all invariant operators of order ≤ 20 , the complete list is not that interesting, we think, unlike the indecomposable operators. We illustrate the answer with a part of the complete list.

Proof. Computer-aided study with the aid of Grozman’s *Mathematica*-based code *SuperLie*, see [Gr]: (1) We fix $\deg D \in \{1, 2, \dots, 20\}$ and $p \in \{2, 3, 5, \dots, 19\}$. (2) For this $\deg D$ and p , we look for which \underline{N} and (a, b) there is a $\mathbf{vect}(1; \underline{N})$ -invariant operator. \square

- *Order 1 operators* are the same as for $p = 0$, but for $p = 2$ we have

(19)	(a, b)	$D(f, g)$	comment
	$(1, b)$ and $(a, 1)$	$\int \otimes \text{id}: f'g \text{ and } fg'$	for $\underline{N} = 1$
	(a, b)	$P_{00}: \alpha f'g + \beta fg'$	for any $\alpha, \beta \in \mathbb{K}$ and $\underline{N} = 1$
	$(0, 0)$	$P_{00}: \alpha f'g + \beta fg'$	for any $\alpha, \beta \in \mathbb{K}$
	(a, b) $(1, 1)$	$\{\cdot, \cdot\}_{P.B.}: = afg' + bfg'$ $\{\cdot, \cdot\}: = fg' + f'g$	Poisson bracket contact bracket of generating functions

• *Order 2 operators* are the same as for $p = 0$, but the condition $a + b + 1 = 0$ is to be replaced with $a + b + 1 = 0 \pmod{p}$ and for $p = 2$ we have

(20)	(a, b)	$D(f, g)$	comment
	(a, b)	$f'g'$	for $\underline{N} = 1$; if $(a, b) = (1, 1)$, then $D = \int \otimes \int$
	$(0, b)$	$f'g' + bf''g$	$\{df, g(du)^b\}_{P.B.}$
	$(a, 1 + a)$	$afg'' + f'g' + (a + 1)f''g$	$\{f(du)^a, dg(du)^{1+a}\}_{P.B.}$
	(a, b)	$ag''f + f'g' + bf''g$	generalizes the above

whereas for $p = 3$ we have

(21)	(a, b)	$D(f, g)$	comment
	(a, b) for $a, b \neq 0, 1$	$J_{a,b}^2(f, g)$	for $\underline{N} = 1$
	$(1, b)$	$\int \otimes \text{id}(f, g) = f''g$	for $\underline{N} = 1$
	$(0, b)$	$f'g' - bf g''$	for $\underline{N} = 1$
	$(a, -1 - a)$	$\{f, g'\}' = afg'' + (1 - a)f'g' + (a + 1)f''g$	
	$(1, 1)$	$Bj_{3,1}(f, g)$	
	$(a, 0)$	$\{f, g'\} = -afg'' + f'g'$	

• *Order 3 operators* are the same as for $p = 0$, but the Grozman operator Gz does not survive for $p = 2$ and 3. For $p = 2$, we additionally have (this might be unclear; in fact, for $(a, b) = (1, 1)$, we have $\int \otimes \text{id}$ and $\text{id} \otimes \int$ and $Bj_{2,2} = \int \otimes \text{id} - \text{id} \otimes \int$):

(22)	(a, b)	$D(f, g)$	comment
	$(1, b)$ for $b \neq 0$	$\int \otimes \text{id}(f, g) = f'''g$	for $\underline{N} = 2$
	$(1, 0)$	$Bj_{2,2}^*, \int \otimes \text{id}$	for $\underline{N} = 2$
	(a, b) for $a, b \neq 0, 1$	$J_{a,b}^3(f, g)$	for $\underline{N} = 2$

• *Order 4 operators*: no operators if $p > 5$; for $p \leq 5$, we have

(23)	(a, b)	$D(f, g)$	p	comments
	$(0, 0)$	$Gz_{2,2}(df, dg) = f'''g' + f'g''' + f''g''$	2	$\underline{N} = 2$
	$(1, 0)$	$Gz_{2,2}^*(df, dg) = fg^{(4)} + f'''g'$	2	$\underline{N} = 2$
	$(0, 1)$	$Gz_{2,2}^{*2}(df, dg) = f^{(4)}g + f'g'''$	2	$\underline{N} = 2$
	$(1, 1)$	$\int \otimes \int = f''g''$	3	for $\underline{N} = 1$
	$(0, 0)$	$Bj_{3,1} \circ (df, dg)$	3	
	$(1, 1)$	$Bj_{5,1}(f, g)$	5	
	$(1, 0)$	$Bj_{5,1}^{*2}(f, g) = fg^{(4)} - f'''g' - f'g''' + f''g''$	5	
	$(0, 1)$	$Bj_{5,1}^*(f, g) = f^{(4)}g - f'''g' + f''g'' - f'g'''$	5	
	$(1, b), b \neq 0$	$\int \otimes \text{id}: f^{(4)}g$	5	for $\underline{N} = 1$
	$(0, 1)$	$\text{id} \otimes \int, Bj_{5,1}^*$	5	for $\underline{N} = 1$
	$(a, 1 - a), a \neq 0, 1$	$L_{5,1}(f, g) = f^{(4)}g + fg^{(4)} - f'''g' - f'g''' + f''g''$	5	

• *Order 5 operators*: no invariant operators if $p > 7$; for $p \leq 7$, we have

(24)	(a, b)	$D(f, g)$	p	comments
	$(0, 0)$	$Bj_{2,2}(df, dg) = f'g^{(4)} + f^{(4)}g'$	2	
	$(0, 0)$	$Gz_{5,1}(df, dg) = f^{(4)}g' - g^{(4)}f' + f''g''' - f'''g''$	5	
	$(0, 1)$	$Gz_{5,1}^{*2}(df, dg) = f^{(5)}g - f'g^{(4)}$	5	
	$(1, 0)$	$Gz_{5,1}^*(df, dg) = fg^{(5)} - f^{(4)}g'$	5	
	$(1, 1)$	$Gz_{7,1}(f, g)$	7	

- *Order 6 operators*: no invariant operators if $p > 7$; for $p \leq 7$, we have

(25)

(a, b)	$D(f, g)$	p	comment
$(1, 1)$	$\int \otimes \int$	2	for $\underline{N} = 2$
$(1, 1)$	$Gz_{2,3}(f, g)$	2	for $\underline{N} > 2$
$(0, 0)$	$Bj_{5,1}(df, dg) = f^{(5)}g' - f'g^{(5)}$	5	
$(0, 1)$	$Bj_{7,1}^{*1}(f, g)$	7	
$(1, 1)$	$Bj_{7,1}(f, g)$	7	
$(1, 0)$	$Bj_{7,1}^{*2}(f, g)$	7	
$(1, b)$ for $b \neq 0$	$\int \otimes \text{id}$	7	for $\underline{N} = 1$
$(1, 0)$	$\int \otimes \text{id}, Bj_{7,1}^{*2}$	7	for $\underline{N} = 1$
$(0, 1)$	$Bj_{7,1}^{*1}, \text{id} \otimes \int$	7	for $\underline{N} = 1$
$(1, 1)$	$Bj_{7,1}(f, g)$	7	for $\underline{N} = 1$
$(a, 1 - a), a \neq 0, 1$	$L_{7,1}(f, g)$	7	

- *Order 7 operators*: no invariant operators if $p > 7$; for $p \leq 7$, we have

(26)

(a, b)	$D(f, g)$	p	comment
$(1, 0)$	$Bj_{2,3}^{*2}(f, g)$	2	
$(1, 1)$	$Bj_{2,3}(f, g)$	2	
$(1, b)$ for $b \neq 0$	$\int \otimes \text{id}$	2	$N = 3$
$(1, 0)$	$\int \otimes \text{id}, Bj_{2,3}^{*2}$	2	$N = 3$
$(a, 1 - a), a \neq 0, 1$	$L_{2,3}(f, g)$	2	$N = 3$
$(1, 1)$	$Gz_{3,2}(f, g)$	3	
$(0, 0)$	$Gz_{7,1}(df, dg)$	7	
$(1, 0)$	$Gz_{7,1}^{*1}(df, dg) = fg^{(7)} - f^{(6)}g'$	7	
$(0, 1)$	$Gz_{7,1}^{*2}(df, dg) = f^{(7)}g - f'g^{(6)}$	7	

Acknowledgements. We are thankful to Grozman for his wonderful code *SuperLie*, see [Gr]. S.B. was partly supported by the grant AD 065 NYUAD.

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